

# On the origin of the decrease in the torsional oscillator period of solid $^4\text{He}$

Z. Nussinov<sup>1</sup>, A. V. Balatsky<sup>2</sup>, M. J. Graf<sup>2</sup>, and S. A. Trugman<sup>2</sup>

<sup>1</sup> *Department of Physics, Washington University, St. Louis, MO 63160 USA and*

<sup>2</sup>*Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA*

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A decrease in the rotational period observed in torsional oscillator measurements was recently taken as a possible indication of a putative supersolid state of helium. We reexamine this interpretation and note that the decrease in the rotation period is also consistent with a solidification of a small liquidlike component into a low-temperature glass. Such a solidification may occur by a low-temperature quench of topological defects (e.g., grain boundaries or dislocations) which we examined in an earlier work. The low-temperature glass can account for not only a monotonic decrease in the rotation period as the temperature is lowered but also explains the peak in the dissipation occurring near the transition point. Unlike the non-classical rotational inertia scenario, which depends on the supersolid fraction, the dependence of the rotational period on external parameters, e.g., the oscillator velocity, provides an alternate interpretation of the oscillator experiments.

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## I. INTRODUCTION

Supersolidity is a unique state of matter that simultaneously displays both superfluidity and crystalline order. Much of our understanding of this exotic state is based on the pioneering work of Penrose and Onsager,<sup>1</sup> Andreev and Lifshitz,<sup>2</sup> Chester and Reatto,<sup>3,4,5</sup> Leggett,<sup>6</sup> and Anderson.<sup>7</sup> Recently, Anderson and coworkers revisited this problem.<sup>8,9</sup> Current developments in this field, presented at a KITP workshop at Santa Barbara, are available online.<sup>10</sup>

$^4\text{He}$  has long been thought to be the most likely candidate for the supersolid state. Torsional oscillator experiments by Kim and Chan<sup>11,12,13,14</sup> generated renewed interest in this possibility.<sup>15,16</sup> In addition to the work of Kim and Chan,<sup>11,12,13,14</sup> there is now an independent confirmation of the anomalous behavior of solid  $^4\text{He}$ , as presented by the groups of Reppy and Shirahama.<sup>17,18,19,20,21</sup> All of these groups use torsional oscillators similar to those employed by Kim and Chan.<sup>11,12,13,14</sup> Rittner and Reppy<sup>18</sup> report history dependence of the signal, when annealing the sample, to the extent of no observation of any mass decoupling in the torsional oscillator experiment. On the other hand, rapid freezing of helium leads to disorder and a drastic increase of the signal.<sup>21</sup> Taken at face value, these torsional oscillator experiments indicate an anomalous mechanical behavior of solid  $^4\text{He}$  at low temperatures. Nevertheless, the connection between the mechanical measurements and the suggested supersolidity remains tenuous. The case for supersolidity entails a certain assumption, which we will discuss at length below, for the interpretation of the oscillator data.

The measurements by Refs. 11,12,13,14,18,19,20,21 monitor the period of torsional oscillators in the presence of an applied torque. The purpose of our work is to examine the mechanical properties of the torsional oscillator. We will only allude to linear response theory and causality (the Kramers-Kronig relations). We show that the change in torsional oscillator period may be triggered by a change in damping and other oscillator parameters. Such a change may result from a low-temperature quench of residual topological defects in the solidification into a glass. Here, we will use the term “liquid-

like” component to describe elastic defects that are not pinned above a glass transition at high temperatures. In Ref. 22, we first explained how a low-temperature pinning of elastic defects (two-level tunneling systems) can account for the reported excess low-temperature specific heat with regards to a perfect Debye crystal. The results of Ref. 22 are at odds with a *simple* uniform supersolid transition, but can also occur in an exotic glassy supersolid state as discussed by Boninsegni et al.<sup>23</sup> or Wu and Phillips<sup>24</sup>. The effects described in this paper may account for the observed change of the torsional oscillator period. The central thesis of our current work is that there is no imperative to adduce a supersolid fraction which depends very sensitively on a host of external parameters in order to explain the reduction in the torsional oscillator period as temperature is lowered. Rather, the observed trends might be a very natural and quite universal outcome of a glass transition (regardless of its classical or quantum nature).

A direct proof of superfluidity would be the observation of persistent current. In this regard, we note a recent experimental search for superflow by Day et. al.,<sup>25,26</sup> who found no mass flow of any kind to very high accuracy. Relying on these results and, most notably, on a new thermodynamic analysis, we recently concluded<sup>22</sup> that the effect first observed<sup>11,12,13</sup> is likely not an intrinsic property of solid  $^4\text{He}$ . This is so because the low-temperature behavior depends critically on the  $^3\text{He}$  concentration, shows annealing and rapid freezing dependence,<sup>18,21</sup> and, most importantly, has a specific heat which is inconsistent with a simple supersolid transition of a 1% condensate fraction.<sup>22</sup> However, our earlier work<sup>22</sup> does not rule out a supersolid fraction of order  $\mathcal{O}(10^{-5})$ .

## II. OUTLINE

Our principal findings are briefly summarized in Sec. (III). To set the stage for these results, we will briefly review, in Sec. (IV), the non-classical rotation inertia (NCRI) effect. We then discuss the current possible ground-breaking implication to the observations of Refs. 11,12,13,14,18,19,20,21 [wherein the change in torsional oscillator period may be triggered by

a supersolid phase]. We then step back in Sec. (V) and analyze the results for a first principle linear response analysis without assuming the NCRI. Consequently, we show how the experimental results are consistent with a change in damping or other oscillator parameters which are brought about by the solidification of a small liquidlike component as the temperature is lowered. In Sec. (VI), we examine the experimental consequences of a uniform supersolid transition and illustrate that a simple supersolid transition cannot account for the peak in the dissipation observed by Ref. 18. We then determine, in Sec. (VII), the angular response for a glass transition, which can account for all of the existing trends in the data, derive in detail our new results on the dissipation and torsional oscillator period for a glass, and suggest an experiment to distinguish between the glass versus supersolid scenario. We conclude in Sec. (VIII) with a brief summary of our results and their relevance to the torsional oscillator data of Refs. 11,12,13,14,18,19,20,21. In the appendix (Sec. A), we illustrate that other, non-glass, response functions can also account for the data without assuming a supersolid transition.

### III. PRINCIPAL RESULTS

We will address the linear response theory of a driven torsional oscillator and derive our key results in Sec. (VII). A general treatment of the problem involves the analysis of possible modes in the vessel which are excited by the external torque and their influence on the torsional oscillator. Depending on their character, such modes can easily be disrupted by the insertion of barriers and lead to a far smaller difference between the response of a liquidlike component and that of a glasslike component. To make the discussion clear, we will focus on a simple form for these modes, which are characteristic of glass formers. Here we will show:

A glass transition, irrespective of whether or not it appears in a normal solid or supersolid, leads to specific testable predictions for the oscillator period and dissipation.<sup>23,24</sup> Unlike a *simple* supersolid transition, i.e., a transition where only a change in the inertia of the oscillator occurs, our key result Eq. (24) predicts that, as the temperature is lowered, two principal trends will be noted: (i) a monotonic decrease in the oscillator period and (ii) a peak in the dissipation. A uniform, mean-field type, supersolid transition cannot account for phenomenon (ii).

It is important to stress, that unlike all of the works to date, we do not assume that only a supersolid can account for the observed decrease in the oscillator period [feature (i)]. Indeed, as we show for the first time, glassy relaxations mandate the current observations of features (i) and (ii). This occurs universally and is independent of the specific microscopic mechanism by which relaxation processes occur or the detailed character of the glass. It is nevertheless encouraging to note that samples in which the most pronounced anomalous behavior is observed contain defects.<sup>21,27</sup>

We considered other (non-glass) “freezing” transitions that can also account for the empirical trends. To make this concrete, in the appendix A, we will detail how a decrease in

damping (as a result of a freezing transition) can also account for the observed tendencies. Unlike for the results discussed above, here we cannot provide a specific functional form to fit to. To provide the simplest qualitative form, we invoke Gaussian distributions for the quantities in question.

Finally, we note that throughout our work, our general form for the angular response allows for a dependence on external parameters such as the initial oscillator velocity. These parameters, e.g. stiffness and density and cell torsional frequency, which parameterize the external torques, are convolved with the intrinsic angular response. A *liquidlike* component in the cell may respond differently to different initial oscillator velocities. Such a dependence is observed experimentally. Aside from other possible effects, external parameters [whether in a supersolid, a glass, or in any other phase] will also always appear via their incorporation in the external torques in the general form for the angular response.

### IV. NON-CLASSICAL ROTATIONAL INERTIA IN A NUTSHELL

The basic idea underlying the torsional oscillator measurements is a test of the non-classical rotational inertia (NCRI) - the inability of a superfluid to rotate under an applied torsional drive. The analog of this experiment on liquid <sup>4</sup>He was performed by Andronikashvili.<sup>28,29</sup> Later Hess and Fairbank<sup>30</sup> verified the existence of the NCRI and of a rotational Meissner effect for superfluid helium, which was predicted by London.<sup>31</sup> As a consequence of a superfluid state, which no longer rotates with the “normal” liquid, the effective moment of inertia decreases as the superfluid fraction  $f_s$  increases

$$I_{eff} = I_{osc} + I_{He}[1 - f_s(T)] \quad (1)$$

which is smaller than the “classical” moment of inertia  $I = I_{osc} + I_{He}$  of the combined oscillator-helium system. For an ideal torsional oscillator of stiffness  $\alpha$ , a measure of the period ( $P = 2\pi\sqrt{I_{eff}/\alpha}$ ) indicates what the superfluid fraction  $f_s$  is. The inferred results for  $f_s$ <sup>30</sup> agree well with the superfluid fraction measured by other probes. The current situation for solid <sup>4</sup>He is far from being as clear cut. The experiments show that the rotational motion under an applied torque becomes more rapid at lower temperature- suggesting an analogue of the NCRI for a *supersolid* phase of helium. In this paper we point out that the measurements are also consistent with less exotic material effects, which can be distinguished by further experiments.

### V. HOW CAN WE SIMPLY UNDERSTAND THE CHANGE IN TORSIONAL OSCILLATOR PERIOD IF IT IS NOT ASSOCIATED WITH A SUPERSOLID?

The NCRI interpretation of the existing data is very simple and suggestive. In what follows, we present a very brief mechanical analysis of the measurements which points to an equally simple explanation. To keep the analysis as general as possible, we rely on first principles alone. The torsional

oscillator experiments measure the susceptibility - they do not directly monitor the moment of inertia of the supersolid. We start by writing down the general equations of motion for a torsional oscillator defined by an angular coordinate  $\theta$ ,

$$[I_{osc} \frac{d^2}{dt^2} + \gamma_{osc} \frac{d}{dt} + \alpha] \theta(t) = \tilde{\tau}_{ext}(t) + \int \tilde{g}(t, t'; T) \theta(t') dt'. \quad (2)$$

Here,  $I_{osc}$  is the moment of inertia of the torsional oscillator,  $\alpha$  is its restoring constant,  $\gamma_{osc}$  is the dissipative coefficient of the oscillator,  $\tilde{g}$  arises from the back action of solid  $^4\text{He}$  (it plays the role of a “polarization”), and  $\tau_{ext}$  is the externally imposed torque. In general,  $\tilde{g}$  is temperature ( $T$ ) dependent. The oscillator angular coordinate  $\tilde{\theta}(t)$  is a convolution of the applied external torque  $\tilde{\tau}_{ex}$  and the response function  $\tilde{\chi}(t, t')$ ,

$$\tilde{\theta}(t) = \int dt' \tilde{\chi}(t, t') \tilde{\tau}_{ext}(t'). \quad (3)$$

Causality demands that  $\tilde{\chi}(t < t') = 0$ . This implies that  $\tilde{\chi}(t, t') = \theta(t - t') \tilde{\chi}(t, t')$ ; under a Fourier transformation, this leads to the Kramers-Kronig relations which we will briefly touch on later. As in any time translationally invariant system, the Fourier amplitude of the angular response of the torsion oscillator is a product

$$\theta(\omega) = \chi(\omega) \tau_{ext}(\omega), \quad (4)$$

with  $\chi = \chi_1 + i\chi_2$  the angular susceptibility and  $\tau_{ext}$  the external torque in Fourier space. For the simple torsional oscillator,

$$\chi^{-1}(\omega) = [\alpha - i\gamma_{osc}\omega - I_{osc}\omega^2 - g(\omega; T)]. \quad (5)$$

Here,  $g(\omega, T)$  is the Fourier transform of the real time  $\tilde{g}$  of Eq. (2). For an ideal normal solid, with a moment of inertia equal to  $I_{ns}$ , which rotates with the oscillator, the back action  $g(\omega; T)$  is

$$g_{ss}(\omega; T) = i\gamma_{He}\omega + I_{ns}(T)\omega^2 \quad (6)$$

with  $I_{ns}(T)$  the normal component of solid  $^4\text{He}$ , which varies with temperature and the dissipation  $\gamma_{He}$  is constant. The moment of inertia contribution of the helium is orders of magnitude smaller than that of the empty vessel - the changes observed in the period are very small and are a remarkable experimental triumph. The subscript in  $g_{ss}$  refers to the supersolid interpretation of the results - only a fraction of solid helium (the “normal part”) rotates in unison with the oscillator. Eq. (6) is consistent with the NCRI form of Eq. (1). However, we do not need to impose this form on the existing data. Current experiments measure the oscillator period  $P = 2\pi/\omega_0$  with  $\omega_0$  the real part of the solution of

$$\chi^{-1}(\omega_0) = 0 \quad (7)$$

at fixed  $T$ .<sup>32</sup> For example, a decrease in the dissipative component  $\gamma_{He}$  in the back action  $g(\omega; T)$  with

$$g_{diss}(\omega; T) = i\gamma_{He}(T)\omega + I_{ns}\omega^2, \quad (8)$$

as the temperature is lowered, will also lead to a shorter rotation period.

This observation offers a qualitatively different interpretation of the experimental results<sup>11,12,13,14,18,19,20,21</sup> with no need to invoke supersolidity. Note that *no* superfluid phase is needed to explain the data and there is *no* NCRI effect; the moment of inertia  $I_{ns}$  is temperature independent.

The origin of the subscript in Eq. (8) alludes to a scenario wherein a higher temperature mobile liquidlike component of the sample “freezes” at lower temperatures leading to a decrease in the dissipation  $\gamma$ . In the appendix, we examine in detail the consequences of Eq. (8). To make our discussion more lucid, we remark that a qualitatively similar effect appears in the rotation of hard versus soft boiled eggs. The solid hard boiled egg rotates faster than the, liquidlike, soft boiled egg.

The measured decrease in the rotation period found<sup>11,12,13,14,18,19,20,21</sup> only implies a crossover in  $\chi$  (and a constraint on  $g$ ). As the real and imaginary parts of  $\chi$  are related by the well known Kramers-Kronig (KK) relations,

$$\begin{aligned} \chi_1(\omega) &= \frac{2}{\pi} P \int_0^\infty d\omega' \frac{\omega' \chi_2(\omega')}{\omega'^2 - \omega^2} \\ \chi_2(\omega) &= -\frac{2\omega}{\pi} P \int_0^\infty d\omega' \frac{\chi_1(\omega')}{\omega'^2 - \omega^2}, \end{aligned} \quad (9)$$

an enhanced decrease in  $\chi_1(\omega)$  often appears with a pronounced peak in  $\chi_2(\omega)$ . A nonvanishing  $\chi_2(\omega)$  at finite frequency mandates dissipation. We expect the data to indicate an increase in dissipation concurrent with the reduction of torsional oscillator period. Whether or not experimental data adhere to the KK relations is a powerful check that needs to be done.

In the sections that follow, we explain in some depth how the supersolid and glass pictures can both account for the decrease in the rotational period when the temperature is lowered. We further examine the dissipation.

## VI. A SUPERSOLID ORIGIN

We now determine the experimental consequences of a simple supersolid transition in which the moment of inertia follows Eq. (1) (the well known NCRI effect). Here, the response is given by Eq. (6). We show that while a mean-field type NCRI effect can, as is well known, account for the decrease of the rotation period it cannot account for the peak in the dissipation  $Q^{-1}$  observed by Ref. 18.

We start by reviewing results for an underdamped harmonic oscillator. The temperature dependent period of a general damped oscillator of combined effective moment of inertia  $I_{eff} = I_{osc} + I_{ns}$ ,<sup>33</sup> stiffness  $\alpha$ , and dissipation  $\gamma$  is

$$\theta(t) = \text{Re}\{A \exp[-i\omega_0 t - \kappa t]\} \quad (10)$$

with  $A$  a complex amplitude. For an underdamped oscillator,

the period

$$P \equiv \frac{2\pi}{\omega_0} = \frac{4\pi I_{eff}(T)}{\sqrt{4\alpha I_{eff}(T) - \gamma^2(T)}}, \quad (11)$$

and the damping rate of the oscillation amplitude

$$\kappa(T) = \frac{\gamma}{2I_{eff}}. \quad (12)$$

The “quality factor”  $Q$ , which monitors the number of oscillations required for a system to have its energy drop by a factor of  $e^{2\pi}$ , is

$$Q = \frac{\sqrt{\alpha I_{eff}}}{\gamma}. \quad (13)$$

We can rewrite the changes in oscillator period as result of small changes in  $\gamma, I_{eff}$  as

$$\begin{aligned} \delta P = & \left[ \left( \frac{4\pi}{\sqrt{4\alpha I_{eff} - \gamma^2}} - \frac{8\pi I_{eff}\alpha}{(4\alpha I_{eff} - \gamma^2)^{3/2}} \right) \delta I_{eff} \right. \\ & \left. + \frac{4\pi\gamma I_{eff}}{(4\alpha I_{eff} - \gamma^2)^{3/2}} \delta\gamma - \frac{2\pi I_{eff}^2}{(4\alpha I_{eff} - \gamma^2)^{3/2}} \delta\alpha \right]. \quad (14) \end{aligned}$$

We see from Eq. (14), for damping  $\gamma < \sqrt{2I_{eff}\alpha}$ , that the observed decrease in the rotation period, i.e.,  $\delta P < 0$ ] as the temperature is decreased may be explained by either (1) a purported supersolid transition with  $g = g_{ss}$ , where  $\delta I_{eff} < 0$  and  $\delta\gamma = 0$ ; (2) a (non-glass) freezing transition with a simplified  $g = g_{diss}$ , where  $\gamma_{He}$  becomes smaller as the temperature is decreased, i.e.,  $\delta\gamma < 0$  and  $\delta I_{eff} = 0$ , for more details see the appendix A; (3) a glass transition with an effective increase in the stiffness constant, where  $\delta\alpha > 0$  as temperature is lowered. The glass response to be studied later in Sec. VII emulates, in part, the features of (3).

First, we will consider a supersolid in which  $g = g_{ss}$  of Eq. (6),  $\gamma = \gamma_{osc} + \gamma_{He}$  is fixed and  $I_{eff}(T)$  varies with temperature. Later on, we will comment on the effect of a drop in dissipation expected to accompany the supersolid state. For  $g = g_{ss}$ , we can deduce  $I_{eff}(T)$  and then predict the damping rate  $\kappa_{ss}(T)$ . In this case, the damping rate

$$\begin{aligned} \kappa_{ss} &= \frac{\gamma_{osc} + \gamma_{He}}{2(I_{osc} + I_{ns}(T))} \\ &= \frac{4\pi^2\gamma}{\alpha P^2 + \sqrt{(\alpha P^2)^2 - 4\pi^2\gamma^2 P^2}}, \quad (15) \end{aligned}$$

or, equivalently,

$$Q_{ss} = \sqrt{\frac{\alpha^2 P^2 + \sqrt{\alpha^4 P^4 - 4\pi^2\alpha^2\gamma^2 P^2}}{8\pi^2\gamma^2}}. \quad (16)$$

In Eqns. (15, 16), only the period  $P$  varies with temperature. In this simple supersolid picture, the  $Q$  factor is monotonic in temperature. If  $I_{eff}$  decreases as the temperature is lowered, then the  $Q$  factor will decrease monotonically as the temperature is lowered.

A simple supersolid transition is inconsistent with the data<sup>18</sup> where a pronounced peak in  $Q^{-1}$  is seen near the transition. A concurrent monotonic drop in the dissipation is expected to accompany the supersolid transition. If the drop in the dissipation  $\gamma$  is proportional to the drop in the effective moment of inertia then  $Q^{-1}$  will still be monotonic. Here,  $Q^{-1}$  may increase as  $T$  is lowered. Such an amended simple supersolid picture is still inconsistent with the observed peak of  $Q^{-1}$ .<sup>18</sup> Obviously, a nonuniform change in both  $\gamma(T)$  and  $I_{eff}(T)$  can be engineered to account for the data. Similarly, a broadening of the transition parameters, similar to that discussed in the appendix, can lead to a peak in  $Q^{-1}$ .

## VII. A GLASS ORIGIN

We next illustrate how a glass transition can account for the observed data. A glass transition will not only give rise to a decrease in the period but, unlike the supersolid scenario, will also lead to a peak in the dissipation observed by Ref. 18.

The external torque is the derivative of the total angular momentum,

$$L(t) = \frac{d}{dt} \int d^3x \rho(\vec{x}) r^2 \frac{d}{dt} \theta(\vec{x}) \quad (17)$$

where  $r$  is the distance to the axis of rotation,  $\rho(\vec{x})$  is the mass density and  $\frac{d}{dt} \theta(\vec{x})$  the local angular velocity about the axis of rotation. The experimentally measured quantity is the angular motion of the torsional oscillator - not that of the bulk helium, which is enclosed in it. Ab initio, we cannot assume that the medium moves as one rigid body. Similar to a vessel partially filled with a liquid component, e.g. a soft boiled egg, an initial imparted external torque can lead to a differential rotation between the outer torsional oscillator and a liquid within it. If the liquid “freezes” into a glass the medium will move with greater uniformity and speed. This may lead to an effect similar to that of the NCRI, although its origin is completely different. Furthermore, as a function of initial rotation speed, the variance between the response of “hard” glasslike and “soft” liquidlike media changes as temperature is lowered. Such a difference is indeed observed in the torsional oscillator experiments on <sup>4</sup>He. Our explanations do not require the supersolid fraction to depend on the rotation speed in order to account for the data.

In what follows, we will analyze the effective equation of motion for the torsional oscillator. Excited modes within the medium in its liquid phase can lead to additional dissipative torques acting on the torsional oscillator. These modes can become far slower and effectively disappear as the system freezes into a glass state. Depending on the character of these modes, the insertion of a barrier may or may not lead to a change in the angular motion of the torsional oscillator. For modes corresponding to internal flows around the axis of rotation, such barriers will disturb the liquidlike modes. Such constrictions will lead to a far smaller difference between the period of rotation of the glass and the period of rotation with liquidlike component of <sup>4</sup>He, if the modes are local.



Henceforth, we will focus on conventional forms for glasslike relaxations which might be of use for fitting data. Glasses are characterized by universal response functions over a wide range of frequencies and temperatures. We assume that the additional dissipative modes triggered by rotation have a similar form. To be specific, we will assume that  $g(\omega; T)$  obeys the simple dependence as seen in the far better measured dielectric response functions (where it reflects the polarization) of glasses. It is given by a Dyson-like equation, which has been invoked in Eq. (5),

$$\begin{aligned}\chi &= \chi_0 + \chi_0 g_{gl} \chi_0 + \dots = \chi_0 [1 - g_{gl} \chi_0]^{-1}, \\ \chi^{-1} &= \chi_0^{-1} - g_{gl}.\end{aligned}\quad (18)$$

Here,

$$\chi_0^{-1} = [\alpha - i\gamma\omega - I_{eff}\omega^2] \quad (19)$$

is the inverse susceptibility of the underdamped system formed by the torsional oscillator chassis and normal solid, while  $g_{gl}$  is the response function of the overdamped glass component. Physically, the first term ( $\chi_0$ ) in Eq. (18) corresponds to the response of the bare oscillator to an applied torque, the possible excitation (by the rotating oscillator) on a transient mode in the liquid/glass medium, which then acts back on the torsional oscillator at a later time ( $\chi_0 g_{gl} \chi_0$ ), and all similar higher order processes. Employing the standard overdamped form for glass response functions<sup>34,35,36</sup>

$$g_{gl} = g_0 [1 - i\omega s]^{-\beta}, \quad (20)$$

we will be able to provide an explicit form for  $\chi^{-1}$  and thus for all periods and dissipation as a function of temperature.

In Eq. (20),  $s$  is the characteristic equilibration time of the liquidlike component when it is perturbed. This time  $s$  diverges in the glass phase (at temperatures  $T < T_0$ ) and is nearly vanishing at very high temperatures. Similarly,  $0 < \beta \leq 1$  is an exponent determining the distribution of local relaxation processes.  $\beta$  corresponds to a stretched exponential exponent in real time. For smaller value of  $\beta$ , the distribution of local relaxation frequencies about the dominant process at  $\omega_g = 1/s$  is more smeared out. If  $\beta = 1$ , then we will have a single overdamped oscillator of damping time  $s$ - Eq. (20) then corresponds to the response function of an oscillator (e.g. Eq. (19)) with no quadratic-in- $\omega$  (inertia) term.

The mobile liquidlike component of the sample “freezes” at lower temperatures into a solid glass. This, e.g., can be triggered by the quenching of dislocations or other defects at low temperatures. In Ref. 27, marked superfluid-like behavior was indeed observed only in samples which contained grain boundaries - no superfluid behavior was noted in crystals of high quality.

This inverse susceptibility of the glass can be approximated by a stretched set of overdamped oscillators. This is the physical content of Davidson-Cole form of Eq. (20). The Davidson-Cole form<sup>34,35,36</sup> fits the dielectric response data of glasses very well. More sophisticated nearly universal response functions, which may improve on the analysis that we give here, were reported in Refs. 37,38. In what follows, we will embark on an initial Davidson-Cole analysis of the  $^4\text{He}$  data. If

we invoke the Davidson-Cole<sup>34,35,36</sup> form for  $g_{gl}$  (Eq. (20)) in the sum of all possible intermediate contributions of the glass to the total response of the oscillator, then, from Eq. (18),

$$\chi^{-1}(\omega) = [\alpha - i\gamma\omega - I_{eff}\omega^2 - \frac{g_0}{(1 - i\omega s)^\beta}]. \quad (21)$$

It is easy to verify, by employing Eqns. (11, 12, 13), that for very high  $T$  (where  $s$  is small) Eq. (21) predicts a period

$$P = \frac{4\pi I_{eff}}{\sqrt{4[\alpha - g_0]I_{eff} - (\gamma + g_0\beta s)^2}}, \quad (22)$$

and a Q-factor

$$Q = \frac{\sqrt{(\alpha - g_0)I_{eff}}}{\gamma + g_0\beta s}. \quad (23)$$

Similarly, at very low  $T$  (where  $s$  diverges), the corresponding quantities are given by Eqns. (11, 12, 13) with no change in the parameters given therein. A comparison of the two limits of high ( $s \rightarrow 0$ ) and low ( $s \rightarrow \infty$ ) temperatures reveals that the  $Q$  factor is higher at low  $T$  while the period  $P$  is higher at very high temperatures  $T$ .

At intermediate temperatures (where  $s$  is of the order of the period of the underdamped oscillator), the angle is a superposition of a many modes stretched about  $\omega = 1/s$ . From Eqns. (2, 4),

$$\begin{aligned}\theta(t) &= \frac{1}{2\pi} \int d\omega [\alpha - i\gamma\omega - I_{eff}\omega^2 \\ &\quad - \frac{g_0}{(1 - i\omega s)^\beta}]^{-1} \int dt' \tau_{ext}(t') \exp[-i\omega(t - t')].\end{aligned} \quad (24)$$

Eq. (24) represents the final form for the angular evolution for all  $T$ . Here,  $\tau_{ext}(t')$  is the value of the external torque at time  $t'$ . For a delta function torque,  $\tau_{ext}(t') = A\delta(t')$ , the last integral simplifies. Within our general formulation, and in Eq. (24) in particular, the observed dependence of the period on the external parameters is perhaps not as surprising as demanding a superfluid fraction strongly depend on the oscillation velocity: For example, different initial velocities would correspond to different values for the internal integral  $\int dt' \tau_{ext}(t') \exp[-i\omega(t - t')]$ . This integral would multiply the total susceptibility in all cases [whether we have a glass transition, a supersolid transition, or any other transitions]. In a treatment which is more detailed than the one which we outline here, the transient modes and the consequent form of the response function  $g$  would also depend on external parameters such as initial rotation velocity.

We anticipate that when the relaxation time is similar to the period of the underdamped oscillator, the dissipation will be maximal. This will lead to the largest magnitude in the imaginary component in the singularities of  $\chi^{-1}$ . Here, the glassy components respond with the same frequency as the “normal” component. At both much higher or much lower temperatures, they merely change the net effective spring constant but do not lead to additional modes which closely interfere with the oscillations of the “normal” (non-glass) part. This leads to a strong decoherence and to a sizable energy loss.

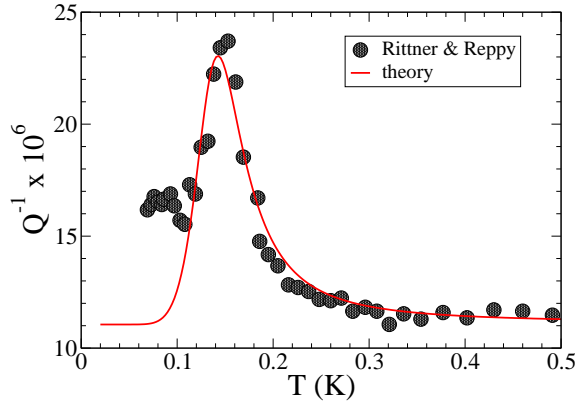


FIG. 1: (Online color) The dissipation (inverse of the quality factor) is shown as a function of temperature for solid  $^4\text{He}$  at 27 bar. The circles are data from Rittner and Reppy<sup>18</sup> and the line is our fit above  $T > 0.1$  K for a simple glass model, Eqns. (25, 26), with the functional form  $Q^{-1} = As[1 + (2\pi f_0 s)^2]^{-1} + Q_{\infty}^{-1}$ . The fit parameters are  $A = 0.02776 \text{ sec}^{-1}$ , oscillator frequency  $f_0 = 184.2313 \text{ Hz}$ , asymptotic inverse quality factor  $Q_{\infty}^{-1} = 11.055 \cdot 10^{-6}$ , glass relaxation time  $s = 1.3537e^{\Delta/k_B T} \mu\text{s}$ , and activation barrier  $\Delta/k_B = 0.9202 \text{ K}$ .

We can examine the  $T < T_0$  (the glass phase) and the  $T = \infty$  limits and perturbations about them when no dissipation is present-  $\gamma = 0$ . In these limits, we have ideal harmonic oscillators with no dissipation and consequently  $Q^{-1} = 0$ . When we expand about the  $T = \infty$  limit and allow for a small  $s$ , then as seen from Eq. (23),  $Q^{-1}$  becomes small but finite (nonzero). Similarly, when  $T = T_0^+$ , where  $s$  is extremely large but finite,  $\chi$  has singularities off the real  $\omega$  axis. In Eq. (24), this small imaginary component now leads to a small dissipation.

This can be made precise for  $\beta = 1$  where the poles are determined by the roots of a cubic equation in  $\omega$ . Here, we can simply look for the largest magnitude of the imaginary part of all of the poles and see when it is maximal as a function of  $s$ . A larger imaginary part implies a shorter decay time and a smaller value of  $Q^{-1}$ . The exact expressions are not illuminating. The asymptotic corrections are, however, very transparent. As can be seen from Eq. (21), for an ideal dissipationless oscillator ( $\gamma = 0$ ),  $\omega_0 = \sqrt{\alpha/I_{eff}}$  is the pole of  $\chi(\omega)$  in the low-temperature limit where  $s \rightarrow \infty$ . If we expand about this pole,  $\omega = \omega_0 + ix + y$ , then we will find that the two dominant poles attain an imaginary component of magnitude

$$|x| = \left[ \frac{g_0}{2I_{eff}} \right] \frac{s}{1 + (\omega_0 s)^2}. \quad (25)$$

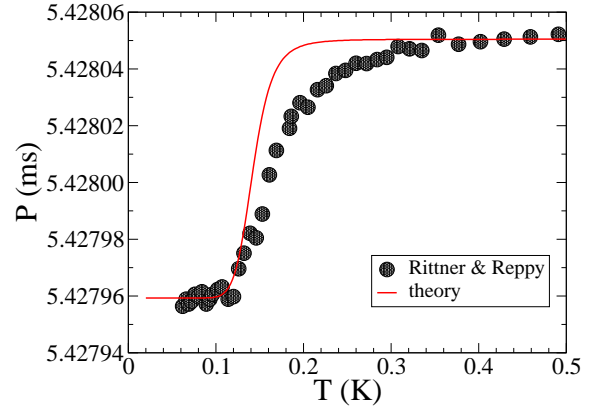


FIG. 2: (Online color) The relaxation period is shown as a function of temperature for solid  $^4\text{He}$  at 27 bar. The circles are data from Rittner and Reppy<sup>18</sup> and the line is our simple glass model, Eq. (27), with the functional form of Eq. (28),  $P = (f_0 - Bf_0^{-1}[1 + (2\pi f_0 s)^2]^{-1})^{-1}$ . Here the *only* adjustable parameter is  $B = 0.57 \text{ sec}^{-2}$ , otherwise same parameters as in Fig. 1.

This Lorentzian peaks, as in Fig. 1, when  $s = 1/\omega_0$ . At this value of  $s$ , the poles have a maximal imaginary component and the dissipation is the largest. In the real dissipative oscillator (with small  $\gamma > 0$ ), the inverse  $Q$  value is given by

$$Q^{-1} = 2|x| \sqrt{\frac{I_{eff}}{\alpha}} + Q_{\infty}^{-1} = \frac{1}{\sqrt{\alpha I_{eff}}} \left[ \frac{g_0 s}{1 + (\omega_0 s)^2} + \gamma \right] \quad (26)$$

with  $Q_{\infty}^{-1} = \frac{\gamma}{\sqrt{\alpha I_{eff}}}$  the linear term in  $\gamma$  (see, e.g., Eq. (13)). By contrast, the correction to the absolute values of the real

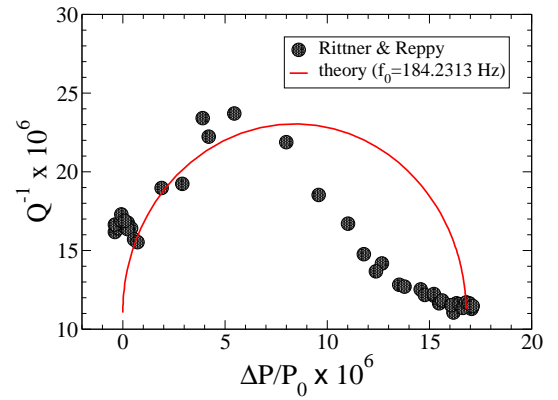


FIG. 3: (Online color) The dissipation-period plot at fixed frequency  $f_0 = P_0^{-1}$ . The curves of Figs. 1 and 2 are replotted against each other. The skewness of the experimental data (circles) compared to the semi-circle line (theory) is a well-known consequence of a *real* glass exponent  $\beta < 1$  [see Eqs.(20, 21)], which for simplicity was set to  $\beta = 1$  in our analysis.

part of the poles is

$$y = -\left[\frac{g_0}{2I_{eff}\omega_0}\right] \frac{1}{1 + (\omega_0 s)^2}. \quad (27)$$

The function  $y$  is a monotonic function of  $s$  and thus of temperature. The period

$$P = \frac{2\pi}{\omega_0 + y} \quad (28)$$

decreases monotonically when the temperature is lowered, see Fig. 2.

The correction for the period due to dissipation is quadratic in  $\gamma$ . This will lead to (i) a monotonic decrease in the period as the temperature is lowered [the increase of  $y$  with lower  $T$  (higher  $s$ )] concurrent with (ii) a peak in the dissipation  $Q^{-1}$  at a temperature  $T^*$ , for which  $s(T^*) = 1/\omega_0$  (seen from the Lorentzian peak for  $|x|$ ), and not to be confused with the ideal glass transition  $T_0$ . The inclusion of finite dissipation ( $\gamma \neq 0$ ) does not modify these trends. Generally, in glasses,  $s$  follows the Vogel-Fulcher-Tamman (VFT) form

$$s \simeq s_0 \exp[DT_0/(T - T_0)] \quad (T > T_0). \quad (29)$$

Here,  $T_0$  is the temperature at which an ideal glass transition occurs. At temperatures  $T < T_0$ , the relaxation time is assumed to be infinite. In some glasses,  $s$  may diverge algebraically in  $(T - T_0)$ . It is important to stress that, in principle, this may even occur if there is no finite temperature glass transition at all and we have a simple activated form with  $T_0 = 0$  in Eq. (29),

$$s = s_0 \exp[\Delta/k_B T], \quad (30)$$

with  $\Delta$  the activation barrier which replaces the constant ( $DT_0$ ) of Eq. (29). In Figs. 1 and 2, we plot the results of our perturbative calculation, Eqns. (26, 27, 28), along with the activated form of Eq. (30) and compare these to the measured data of Ref. 18. In Fig. 3 we show the dissipation-period plot of the same data at fixed frequency  $f_0$  instead of the conventional Cole plot,<sup>34,35,36</sup> because experimentally neither  $\chi_1(\omega)$ , nor  $\chi_2(\omega)$  are available at fixed  $T$ . Note that for a real glass there are many more parameters than those that we fit with here. These parameters are (i)  $0 < \beta \leq 1$ , which we set to 1 in our equations, and (ii) the ideal glass transition temperature  $T_0 \geq 0$ , which we set to zero in the activated form of Eq. (30). In fact, an exponent  $\beta < 1$  can explain the skewness<sup>34,35,36</sup> of the experimental result in Fig. 3 compared to the semi-circle result of our simplified analysis with  $\beta = 1$  [see Eqs.(25, 26, 27, 28)].

The four relations of Eqns. (11, 13, 22, 23) along with the four measured values [the periods at the high and low temperatures, the  $Q$  factors at high and low temperatures] allow us to determine the four parameters  $\alpha, \gamma, I_{eff}$ , and  $g$ . In fact, due to the scale invariance of the relations, there are only three unknowns (e.g.  $\alpha/I_{eff}, \gamma/I_{eff}$ , and  $g/I_{eff}$ ) from which the fourth value can be predicted and compared to its experimental value. Currently, the available data centers on the region of the transition and it is not certain what the asymptotic low and

high temperature values of the  $Q$  factors ( $Q_0$  for  $T = T_0^+$ ,  $Q_\infty$  for  $T \gg T_0$ ) or the periods ( $P_0$  for  $T = T_0^+$ ,  $P_\infty$  for  $T \gg T_0$ ) are. If we solve for all unknown quantities in terms of  $\{P_0, P_\infty, Q_\infty\}$ , then we have

$$\begin{aligned} \alpha/I_{eff} &= 4\pi^2 (P_0^{-2} - P_\infty^{-2} [4Q_\infty^2 - 1]^{-1}) \approx 4\pi^2/P_0, \\ \gamma/I_{eff} &= 4\pi(Q_\infty P_\infty)^{-1} (4 - Q_\infty^{-2})^{-1/2} \approx 2\pi/(Q_\infty P_\infty), \\ g/I_{eff} &= 4\pi^2 (P_0^{-2} - P_\infty^{-2}) \approx \alpha(1 - P_0^2/P_\infty^2). \end{aligned} \quad (31)$$

The choice for the numerical value of the effective moment of inertia  $I_{eff}$  corresponds to a different choice of units. Neither the precise values of the parameters  $s_0$ ,  $\Delta$ , and  $T_0$  of Eq. (29), nor of  $\beta$  in Eq. (24), change qualitatively the form of the fits that result from Eq. (24). All that matters is that at the temperature of the apparent transition  $T^*$ , the dominant relaxation time

$$\bar{s}(T^*) = 2\pi s(T^*) \simeq P_0, \quad (32)$$

matches with the torsional oscillator period. Putting all of the pieces together, Eq. (24) leads to a monotonic decrease of the rotation period as the temperature is lowered. Hand in hand with that, the dissipation becomes maximal when  $\bar{s}$  is equal to the period of the unperturbed oscillator.

A possible testable consequence of this scenario is an experiment in which a torsional oscillator containing solid  $^4\text{He}$  is set in motion, and finally is abruptly stopped and then let go again. If the sample is first cooled below the glass transition while at rest, then the vessel contains a solid glass and the torsional oscillator will stop perfectly. Because all of the material inside the oscillator will stop in unison with the container. However, if the oscillator is stopped abruptly and let go at temperatures above the glass transition, then the vessel contains a liquidlike component and the torsional oscillator will wobble due to the motion of the residual liquidlike component. Here, the liquidlike component will still move even after the container is stopped. This internal liquidlike motion will lead to a momentum transfer to the outer torsional oscillator once the container is let go. Hence, the oscillator will wobble.

## VIII. CONCLUSIONS

In conclusion, we report on a simple alternate proposal of the origin of the decrease in the torsional oscillator period in solid helium.<sup>11,12,13,14,18,19,20,21</sup> We argue that changes in the oscillation period are triggered by changes in damping and the effective stiffness in the susceptibility and are not due to a supersolid state formation. We point out that the torsional oscillator mechanical properties can be controlled by a number of effects such as damping, stiffness coefficient and moment of inertia changes. The supersolid interpretation, if such a state exists, would lead to changes in moment of inertia. This was the interpretation taken in Refs. 11,12,13,14. On the other hand, low-temperature quenching of residual topological defects in the solidification into a glass can also account for the current observation of a drop in the torsional oscillator period as the temperature is reduced. This effect not only

accounts for the decrease in the rotation period (Fig. 2), but also predicts a peak in the dissipation, see Fig. 1, as indeed observed.<sup>18</sup> Furthermore, our alternate glass scenario can account for the observed dependence of the oscillator dynamics on external parameters.

We have developed a basic understanding of how a simplified response of a glass can account for the essential physics results of the torsional oscillator experiments. Specific heat data further show very strong support for a low-temperature glass phase<sup>22</sup> and effectively rule out a simple supersolid transition by thermodynamic measurements. Based on our analysis, we can make the following specific predictions which conform with the data available to date:

(1) At low temperatures all dynamic observables, e.g.,  $Q^{-1}$  and  $P$ , will collapse on a single curve as a function of temperature, which is independent of the frequency of the torsional oscillator. This collapse follows from Eqs. (18), (19), and (29). It reflects the freezing-out of liquidlike degrees of freedom, where the response is controlled by the *normal* susceptibility  $\chi_0$ . In general, the higher the resonant frequency  $\omega_0$  is, the higher the temperature  $T^*$  at which this freeze out will occur:  $s(T^*) = 1/\omega_0$ , as can be seen from Eq.(29),  $T^* > T_0$ . At temperatures  $T \leq T_0$ , all of the data collected from oscillators with different resonant frequencies, in which all other parameters are held fixed, should collapse onto a single curve.

(2) In all known glasses the exponent  $\beta < 1$  [Eq.(20)] and the glass temperature  $T_0 > 0$  K, [Eq. (29)], while in our analysis we assumed for simplicity and to avoid the use of too many adjustable parameters that  $\beta = 1$  and  $T_0 = 0$  K. For more realistic glass models one would need to relax these constraints. Indeed, Fig.(3) vividly illustrates that better fits may be obtained with values of  $\beta < 1$ . Similarly, all of the available data to date are consistent with values of  $0 < T_0 \lesssim 100$  mK. The more accurate detailed predictions [albeit with more phenomenological glass parameters] follow from Eqs.(21, 24, 29). We leave this analysis for a separate discussion when more data are available.

(3) The blockage of the annulus in the torsional oscillator cell may drastically change the strain field and dynamics inside the cell. Such changes of the boundary conditions can inhibit the presence of various elastic modes relative to those which are present in a system with no blockage. For instance, azimuthal elastic modes which may be present in an “O-type” cylindrical geometry cannot appear in a blocked “C-type” cylindrical configuration. More generally, any modification to the boundary conditions, e.g., oscillator velocity or amplitude, sample surface or roughness, would shift the freezing-out temperature and damping. For instance, large amplitude vibrations may thwart the quench of a liquidlike component into a glass. Such an avoidance of the glass transition will, consequently, lessen the low temperature variations in the oscillator dynamics. All of these effects will necessarily lead to different experimental observables without a need for invoking a supersolid fraction which somehow depends very sensitively on all of these parameters.

As a matter of principle, our glass picture explains the data even if a glass transition accompanies the transition into a supersolid state.<sup>23,24</sup> What we show is that whether or not a

tiny supersolid fraction is present- the changes in the dynamics can be naturally ascribed to a glass transition rather than to a Non Classical Rotational Inertia (NCRI) effect. Our results demonstrate that a low-temperature glass can very naturally account for the current data. What our analysis suggests is whether or not a very small supersolid fraction is present, what triggers the change in period may (very naturally) be a hallmark of the transition into a glassy state. The changes in the torsional oscillator period need not be accounted for by only a non classical rotational inertia effect. A glass transition (whether it is classical or quantum is immaterial on this level of our analysis) may further naturally account for the observed subtle dependence of the effect on external parameters as well as the concurrency of the peak in the damping with the change in oscillator period. It is worth noting that a decrease in the rotational period along with a peak in the dissipation can be triggered by a host of many other related effects - none of which relies on a NCRI effect. As an example, in the appendix we detail an alternate model in which the dissipation decreases by the solidification of liquidlike (topological) defects into a non-glassy low-temperature phase. Similarly, any other effect which leads to a decrease in the oscillator period with decreasing temperature and exhibits a peak in  $Q^{-1}$  can account for the current data.

Finally, an experiment in which a torsional oscillator holding <sup>4</sup>He is first cooled when it is at rest, then set in motion and abruptly stopped and let go, might allow further comparison with the glass origin that we propose here for the anomalous low temperature dynamics. For a glass transition scenario, we expect (1) that at temperatures above the glass transition, the liquidlike component will cause the torsional oscillator to wobble. While (2) the same experiment performed sufficiently below the glass transition temperature, the glass and container will stop in unison without wobbling.

## IX. ACKNOWLEDGMENTS

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## APPENDIX A: AN ALTERNATE EFFECT - DISSIPATION LOWERING UPON FREEZING

Here, we show how the seminal qualitative features of the current available data<sup>11,12,13,14,18,19,20,21</sup> can be explained by not only a glass transition [or even by its more restrictive limiting form, which is given by the simple activated dynamics of Eqns. (20, 30) with  $\beta = 1$  plotted in Figs. 1 and 2), but also by other, more general, “freezing” transitions. Glasses adhere to specific, nearly universal, response functions and relaxation time forms - e.g. that of Eq. (20).<sup>34,35,36</sup> We now consider another “freezing” scenario of (topological) defects



in which, for illustrative purposes, only the dissipation varies with temperature. Upon cooling, the liquidlike regions in the solid "freeze" and lead to a lower average dissipation.

In the dissipative response  $g_{diss}$  of Eq. (8), the dissipation  $\gamma(T) = \gamma_{He}(T) + \gamma_{osc}$  varies with temperature while  $I_{eff} = I_{osc} + I_{ns}$  is fixed. Thus, a measured value of the period can be used to compute  $\gamma(T)$  and then predict the amplitude damping rate  $\kappa_{diss}(T)$  and compare it to experiment. Here,

$$\begin{aligned} \kappa_{diss}(T) &= \frac{\gamma_{osc} + \gamma_{He}(T)}{2(I_{osc} + I_{ns})} \\ &= \frac{\sqrt{\alpha I_{eff} P^2 - 4\pi^2 I_{eff}^2}}{I_{eff} P}, \end{aligned} \quad (A1)$$

with  $I_{eff} = I_{osc} + I_{ns}$  now a constant quantity which is independent of the temperature. Similarly, the Q factor (determined by  $\kappa$ ) is

$$Q_{diss} = \frac{P\sqrt{\alpha}}{2\sqrt{\alpha P^2 - 4\pi^2}}. \quad (A2)$$

If this alone is what occurs then at all temperatures, the dissipation rate  $\kappa$  and Q-factor will be related to the period by Eqns. (A1, A2) parameterized by the two constants  $I_{eff}$  and  $\alpha$ . On the right hand sides of Eqns. (A1, A2), only the measured oscillator period  $\kappa$  varies with temperature, as indeed observed.<sup>11,12,13,14,18,19,20,21</sup>

The Q factor monotonically increases as the temperature decreases. This is evident from the final form of Eq. (A2) or even from the trivial Eq. (13). Because  $\gamma$  decreases as the temperature is lowered, an ensuing increase of Q follows.

We will now address the non-monotonic character of the dissipation. This effect is not present in the treatment thus far. The dissipation (probed by  $Q^{-1}$ ) was seen<sup>18</sup> to exhibit a pronounced maximum in nearly the temperature range where the oscillator period shows its most dramatic decrease. Now, near transition points (whether those of a defect freezing or of other origins), fluctuations and consequent dissipation are very often large. Here, we cannot think of only one damped oscillator for either the quenching of defects nor other transition scenario. To make contact with the previous approach, we may divide the system into external torsional oscillators coupled to a multitude of small damped oscillators within the medium. If the medium is homogeneous then the system can be thought of as a single mode oscillator. Otherwise, a more careful analysis is required. In general, we will need to diagonalize the  $N \times N$  matrix  $[\alpha - i\gamma\omega - \omega^2 I]$  with the elements  $I_{ij}$ ,  $\alpha_{ij}$  and  $\gamma_{ij}$  being the parameters appearing in the equations of motion for the angles  $\{\theta_i\}_{i=1}^N$  of different elements of the solid.

In the context of defect freezing, we might anticipate some defects to already be frozen (and lead to a lower value of  $\gamma$  in those volume elements) and for other volume elements to have fewer frozen defects and to correspond to a higher value of  $\gamma$ . This distribution of damping coefficients leads to a solution for  $\theta(t)$  which is a superposition of many modes dispersed in frequency. The broader the frequency dispersion- the larger the incoherence- the smaller the value of the  $\langle\theta^2\rangle$  and that

of the energy. As a result, the  $1/Q$  value which records the damping of the energy may be much higher near the transition.

We next implement the stochastic character of the value of  $\gamma$  for each volume element. For illustrative purposes let us consider a distribution given by

$$P(\gamma) = \frac{1}{\sqrt{2\pi\sigma_\gamma^2}} \exp[-(\gamma - \gamma_0)^2/(2\sigma_\gamma^2)]. \quad (A3)$$

The width  $\sigma_\gamma$  is generally a function of temperature. It is finite within the transition region and tends to zero outside it. Far from the transition, we have a single value of  $\gamma$  [i.e.,  $P(\gamma) = \delta(\gamma - \gamma_0)$ ] and our former analysis applies. Within the transition region  $\gamma_0$  can be read off from the period by the use of Eq. (11).

The angle is now a superposition of all modes. For a delta function torque  $\tau_{ext}(t) = A\delta(t)$ ,

$$\theta(t) = \text{Re}\{A \int_0^\infty d\gamma P(\gamma) \theta_\gamma \exp[-i\omega_0^\gamma t - \kappa^\gamma t]\}, \quad (A4)$$

$g(\gamma)$  the distribution of  $\gamma$  values, which we anticipate to be approximated by Eq. (A3), and with  $\omega_0^\gamma$  and  $r^\gamma$  given by Eqns. (11, 12).

To get an intuitive feeling for the damping triggered by the dispersion, we can expand the argument in Eq. (A4) about  $\gamma_0$  to obtain the original solution (that assuming a single mode defined by a unique  $\gamma_0$ ) multiplied by the approximate damping factor

$$C = \frac{1}{\sqrt{2\pi\sigma_\gamma^2}} \int_{-\infty}^\infty dz \exp[-iatz - \frac{zt}{2I_{eff}} - z^2/(2\sigma_\gamma^2)]. \quad (A5)$$

Here,  $z \equiv (\gamma - \gamma_0)$  and

$$a = \frac{\gamma}{I} \frac{1}{\sqrt{4\alpha I - \gamma^2}}. \quad (A6)$$

Let us first examine an ideal situation where no damping initially exists ( $\kappa^\gamma = 0$ ). Here, Eq. (A5) is none other than the Fourier transform of a Gaussian which is another Gaussian given by a time dependence,  $C = \exp[-\frac{1}{2}a^2\sigma_\gamma^2 t^2]$ . This factor leads to an energy dissipation in time that does not appear for a single mode dispersion: the energy is damped by a factor of  $C^2$  relative to that of the nondispersive system. Dispersion alone can lead to damping. Taking dissipation into account leads to a similar Gaussian factor of

$$C = \exp[-\frac{1}{2}\sigma_\gamma^2 t^2 (a^2 + \frac{1}{4I_{eff}^2})] \quad (A7)$$

for the amplitude along with an additional small correction to the phase terms  $\phi = \omega_0 t \rightarrow \omega_0 t + \frac{\sigma_\gamma^2 a}{2I} t^2$ . This additional phase factor is negligible (over a period of the oscillator) if  $(4\alpha I - \gamma^2)^{3/2} \gg 2\sigma_\gamma^2$ .

The above Gaussian approximation is just a heuristic motivation for visualizing how dispersion lowers the average energy. The precise integral is that of Eq. (A4). If the width

$\sigma(T)$  peaks at the transition temperature then the dissipation is amplified near the transition temperature while, all along, the oscillator period is monotonic in temperature. To con-

clude, we illustrated, once again, how the torsional oscillator data can be accounted for without invoking a supersolid transition.

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  - <sup>32</sup> This form is what immediately transpires for  $\chi(\omega)$  with poles of constant  $|\Re\{\omega\}|$ . When this general form is applied to the standard oscillator, the period is indeed the interval between consecutive local maxima of  $\theta(t)$  [or, equivalently is twice the interval between consecutive zeros of  $\theta(t)$ ].
  - <sup>33</sup> Here and throughout,  $I_{eff}$  is defined by the equations of motion; it is determined by the coefficient of  $\omega^2$  in  $\chi^{-1}$ . Generally,  $I_{eff} = I_{osc} + I_{ns}$  with  $I_{ns}$  the moment of inertia of the normal He solid and  $I_{osc}$  the moment of inertia of the torsional oscillator chassis without the inserted helium. In the presence of a supersolid component,  $I_{eff} < I$ , where  $I$  is the moment of inertia of the entire system. Similar definitions apply to the dissipative coefficient in  $\chi^{-1}$ . With  $\gamma_{osc}$  the dissipative coefficient of the bare torsional oscillator chassis the additional dissipative contribution resulting from the inserted helium is given by  $\gamma_{He} = \gamma - \gamma_{osc}$ .
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